INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 5305–5314 PII: S0305-4470(02)32931-7

Torque exerted by a moving electric charge on a stationary electric charge distribution

Oleg D Jefimenko

Physics Department, West Virginia University, PO Box 6315, Morgantown, WV 26506-6315, USA

E-mail: odjef@wvnvms.wvnet.edu

Received 22 January 2002, in final form 15 April 2002
Published 14 June 2002
Online at stacks.iop.org/JPhysA/35/5305

Abstract

The interaction between a moving electric charge and a stationary electric charge distribution is considered. It is shown that the interaction involves not only an electric attraction or repulsion but also a heretofore unreported electric torque exerted by the moving charge on the stationary charge distribution. The torque is associated with the asymmetry of the electric field of the moving charge and is present even if the stationary charge distribution is highly symmetrical, such as a uniformly charged sphere, for example. As a result of the torque, the stationary charge distribution is set in rotation. The rotating stationary charge distribution creates a magnetic field and an induced electric field that act on the moving charge thus further contributing to the complexity of the interaction. Two types of moving charges are considered: a point charge moving with constant speed along a straight line and a point charge moving with constant speed along a circular orbit. The torques exerted by these charges on stationary charge distributions in the shape of a small circular ring, small disc, and small sphere of uniform charge density are computed and some consequences of these torques are discussed. The possibility of the existence of a similar interaction effect in gravitational systems is also considered.

PACS number: 03.50De

1. Introduction

It is generally believed that a highly symmetrical charge distribution (a spherical charge of uniform charge density, for example) located in an external electric field experiences a force but does not experience a torque[1]. However, as is shown in this paper, even a spherical charge

[1] It is generally assumed that interacting electric charge distributions do not exert torques on each other unless at least one of the charge distributions has a permanent or induced electric dipole moment. See, for example, [1].

of uniform charge density experiences not only a force but also a torque when located in the field of a moving charge.

In the calculations that follow, two types of moving charges are considered: a point charge moving with constant speed along a straight line (constant velocity vector $\boldsymbol{v}$) and a point charge moving with constant speed along a circular orbit.

As is known, the electric field $\boldsymbol{E}$ of a point charge q moving with constant velocity $\boldsymbol{v}$ is represented by the Heaviside's formula [2–5]

$$\boldsymbol{E} = \frac{q(1 - v^2/c^2)}{4\pi\varepsilon_0 r^3[1 - (v^2/c^2)\sin^2\theta]^{3/2}}\boldsymbol{r} \tag{1}$$

where ε_0 is the permittivity of space, c is the velocity of light, $\boldsymbol{r}$ is the radius vector directed from q to the point of observation, r is the magnitude of $\boldsymbol{r}$, v is the magnitude of $\boldsymbol{v}$ and θ is the angle between $\boldsymbol{r}$ and $\boldsymbol{v}$.

The electric field of a point charge q moving with acceleration $\dot{\boldsymbol{v}}$ is represented by the formula [6–8]

$$E = \frac{q}{4\pi\varepsilon_0 r^3(1 - \boldsymbol{r}\cdot\boldsymbol{v}/rc)^3}\left\{\left(\boldsymbol{r} - \frac{r\boldsymbol{v}}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \boldsymbol{r}\times\left[\left(\boldsymbol{r} - \frac{r\boldsymbol{v}}{c}\right)\times\frac{\dot{\boldsymbol{v}}}{c^2}\right]\right\} \tag{2}$$

where the notation is the same as in equation (1), except that $\dot{\boldsymbol{v}}$, $\boldsymbol{r}$ and $\boldsymbol{v}$ are *retarded*, that is, evaluated for the time $t' = t - r/c$, where t is the present time (the time for which $\boldsymbol{E}$ is evaluated). If the charge moves along a circular orbit of radius R, the acceleration is $\dot{\boldsymbol{v}} = (v^2/R^2)\boldsymbol{R}$, where $\boldsymbol{R}$ is directed from q to the centre of the orbit. Therefore for a charge moving along a circular orbit equation (2) becomes

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 r^3(1 - \boldsymbol{r}\cdot\boldsymbol{v}/rc)^3}\left\{\left(\boldsymbol{r} - \frac{r\boldsymbol{v}}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \boldsymbol{r}\times\left[\left(\boldsymbol{r} - \frac{r\boldsymbol{v}}{c}\right)\times\frac{v^2\boldsymbol{R}}{c^2R^2}\right]\right\}. \tag{3}$$

In contrast to the electric field of a stationary point charge, the electric fields represented by equations (1)–(3) are not radially symmetric. As we shall see, it is the asymmetry of these fields that is responsible for the torque and rotation experienced by highly symmetrical charge distributions under the action of these fields.

Because of the complexity of equations (1)–(3), exact analytical calculations of torques exerted on arbitrary charge distributions by point charges moving at arbitrary speeds are hardly possible. Therefore we shall restrict our calculations to the special case of moving point charges whose velocity is considerably smaller than the velocity of light and to the special case of stationary charge distributions whose linear dimensions are considerably smaller than the distance of these charge distributions from the moving charge.

2. Torque due to a point charge moving with constant velocity

Let a negative point charge q move with constant velocity $\boldsymbol{v}$ past a positive spherical charge Q of uniform charge density ρ, and let q and the centre of Q be in a plane normal to the page (figure 1). Consider two points P_1 and P_2 within Q located symmetrically with respect to that plane. According to equation (1), the force $F_1 = E_1\mathrm{d}Q$ acting on the charge element $\mathrm{d}Q$ located at P_1 is larger than the force $F_2 = E_2\mathrm{d}Q$ acting on the charge element $\mathrm{d}Q$ located at P_2 (because $\sin\theta_1$ is larger than $\sin\theta_2$). Therefore the torque with respect to the centre of Q acting on $\mathrm{d}Q$ at P_1 is also larger than the oppositely directed torque with respect to the centre of Q acting on $\mathrm{d}Q$ at P_2. Since the same considerations apply to all such symmetrically located points within Q, the charge Q experiences a net torque with respect to its centre and, as a result, is caused to rotate about its centre (we assume that the mass density of Q is uniform). In particular, for the configuration of q and Q shown in figure 1, Q rotates clockwise.

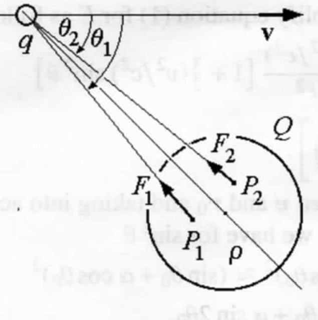

Figure 1. The force F_1 acting on the charge element located at P_1 is larger than the force F_2 acting on an equal charge element located at P_2. Therefore the stationary charge Q experiences a torque causing it to rotate.

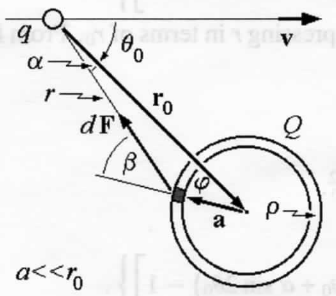

Figure 2. The torque acting on the ring carrying a charge Q is found by integrating the torque acting on the shaded segment of the ring.

As mentioned in the introduction, exact calculations of the torque exerted by point charges moving at arbitrary velocities on stationary charge distributions of arbitrary linear dimensions are difficult. Therefore we shall calculate the torque for the special case of a moving point charge q whose velocity $\boldsymbol{v}$ satisfies the relation $v \ll c$, and, as the stationary charge distribution Q, we shall use a charged ring, disc and sphere whose radius a satisfies the relation $a \ll r_0$, where r_0 is the distance between q and the centre of Q.

2.1. Torque on a uniformly charged ring

Let a negative point charge q move with constant velocity $\boldsymbol{v}$ in the plane of a charged ring of radius a and cross-sectional area S carrying a uniformly distributed positive charge Q of density ρ, as shown in figure 2. Let $\boldsymbol{v}$ satisfy the relation $v \ll c$ and let the radius vector $\boldsymbol{r}_0$ representing the distance from q to the centre of the ring satisfy the relation $a \ll r_0$. The torque $\mathrm{d}\boldsymbol{T}$ with respect to the centre of the ring exerted by q on the charge element $\rho Sa\,\mathrm{d}\phi$ contained in the shaded segment of the ring is then

$$\mathrm{d}\boldsymbol{T} = \rho Sa\,\mathrm{d}\phi(\boldsymbol{a} \times \boldsymbol{E}) = \boldsymbol{k}\rho Sa^2 E \sin\beta\,\mathrm{d}\phi = \boldsymbol{k}\rho Sa^2 E \sin(\phi + \alpha)\,\mathrm{d}\phi \qquad (4)$$

where the angles α, β and ϕ are as shown in figure 2, $\boldsymbol{E}$ is the electric field produced by q at the location of the charge element, E is the magnitude of $\boldsymbol{E}$, and $\boldsymbol{k}$ is a unit vector directed into the page.

Since, by supposition, $v \ll c$, we can simplify equation (1) for E as follows:

$$E = \frac{q(1 - v^2/c^2)}{4\pi\varepsilon_0 r^2[1 - (v^2/c^2)\sin^2\theta]^{3/2}} \approx \frac{q(1 - v^2/c^2)}{4\pi\varepsilon_0 r^2}\left[1 + \tfrac{3}{2}(v^2/c^2)\sin^2\theta\right]$$
$$\approx \frac{q}{4\pi\varepsilon_0 r^2}\left[1 + \frac{v^2}{c^2}\left(\frac{3}{2}\sin^2\theta - 1\right)\right]. \qquad (5)$$

Using $\theta = \theta_0 + \alpha$, where θ_0 is the angle between $\boldsymbol{v}$ and $\boldsymbol{r}_0$ and taking into account that α is a small angle (because, by supposition, $a \ll r_0$), we have for $\sin^2\theta$

$$\sin^2\theta = \sin^2(\theta_0 + \alpha) = (\sin\theta_0\cos\alpha + \sin\alpha\cos\theta_0)^2 \approx (\sin\theta_0 + \alpha\cos\theta_0)^2$$
$$\approx \sin^2\theta_0 + 2\alpha\sin\theta_0\cos\theta_0 = \sin^2\theta_0 + \alpha\sin 2\theta_0. \qquad (6)$$

Equation (5) can therefore be written as

$$E \approx \frac{q}{4\pi\varepsilon_0 r^2}\left\{1 + \frac{v^2}{c^2}\left[\frac{3}{2}\left(\sin^2\theta_0 + \alpha\sin 2\theta_0\right) - 1\right]\right\}. \qquad (7)$$

We can further simplify equation (7) by expressing r in terms of r_0. From figure 2 we see that

$$r^2 = r_0^2 - a^2 - 2ar_0\cos\phi \qquad (8)$$

which, since $a \ll r_0$, can be written as

$$r^2 \approx r_0^2[1 - 2(a/r_0)\cos\phi] \approx r_0^2. \qquad (9)$$

Equation (7) therefore becomes

$$E \approx \frac{q}{4\pi\varepsilon_0 r_0^2}\left\{1 + \frac{v^2}{c^2}\left[\frac{3}{2}\left(\sin^2\theta_0 + \alpha\sin 2\theta_0\right) - 1\right]\right\}. \qquad (10)$$

Now, remembering that α is a small angle, we simplify equation (4) to

$$\mathrm{d}\boldsymbol{T} = \boldsymbol{k}\rho S a^2 E\sin(\phi + \alpha)\,\mathrm{d}\phi \approx \boldsymbol{k}\rho S a^2 E(\sin\phi + \alpha\cos\phi)\,\mathrm{d}\phi. \qquad (11)$$

Substituting E from equation (10), we then have

$$\mathrm{d}\boldsymbol{T} = \frac{\boldsymbol{k}\rho S a^2 q}{4\pi\varepsilon_0 r_0^2}\left\{1 + \frac{v^2}{c^2}\left[\frac{3}{2}(\sin^2\theta_0 + \alpha\sin 2\theta_0) - 1\right]\right\}(\sin\phi + \alpha\cos\phi)\,\mathrm{d}\phi. \qquad (12)$$

Finally, recognizing from figure 2 that $\alpha \approx (a\sin\phi)/r_0$, we obtain

$$\mathrm{d}\boldsymbol{T} = \frac{\boldsymbol{k}\rho S a^2 q}{4\pi\varepsilon_0 r_0^2}\left\{1 + \frac{v^2}{c^2}\left[\frac{3}{2}\left(\sin^2\theta_0 + \frac{a\sin 2\theta_0\sin\phi}{r_0}\right) - 1\right]\right\}\left(\sin\phi + \frac{a\cos\phi\sin\phi}{r_0}\right)\mathrm{d}\phi. \qquad (13)$$

Integrating equation (13) from 0 to 2π, we have

$$\boldsymbol{T} \approx \frac{\boldsymbol{k}\rho S a^2 q}{4\pi\varepsilon_0 r_0^2}\int_0^{2\pi}\left\{1 + \frac{v^2}{c^2}\left[\frac{3}{2}\left(\sin^2\theta_0 + \frac{a\sin 2\theta_0\sin\phi}{r_0}\right) - 1\right]\right\}\left(\sin\phi + \frac{a\cos\phi\sin\phi}{r_0}\right)\mathrm{d}\phi \qquad (14)$$

which gives for the torque acting on the entire ring

$$\boldsymbol{T} \approx \boldsymbol{k}\frac{3\rho S q a^3 v^2}{8\varepsilon_0 r_0^3 c^2}\sin 2\theta_0 \qquad (15)$$

where we have dropped the term with a^2/r_0^2.

Replacing ρ in equation (15) by $Q/2\pi a S$, we obtain the expression for the torque in terms of the charge Q of the ring:

$$\boldsymbol{T} \approx \boldsymbol{k}\frac{3qQa^2v^2}{16\varepsilon_0 r_0^3 c^2}\sin 2\theta_0. \qquad (16)$$

2.2. *Torque on a uniformly charged disc*

We can use equation (15) for finding the torque acting on a small charged disc by considering the ring shown in figure 2 to be a differential element of the disc.

Let the thickness of the disc be τ and let its radius be a. Replacing S in equation (15) by $\tau\,\mathrm{d}x$, replacing a by x and integrating over x from 0 to a, we obtain for the torque acting on the disc

$$\boldsymbol{T} \approx \boldsymbol{k}\frac{3\rho\tau q v^2}{8\varepsilon_0 r_0^3 c^2}\sin 2\theta_0 \int_0^a x^3\,\mathrm{d}x \tag{17}$$

or

$$\boldsymbol{T} \approx \boldsymbol{k}\frac{3\rho\tau a^4 q v^2}{32\varepsilon_0 r_0^3 c^2}\sin 2\theta_0. \tag{18}$$

Replacing ρ in equation (18) by $Q/\pi a^2\tau$, we find the torque acting on the disc in terms of the charge Q of the disc:

$$\boldsymbol{T} \approx \boldsymbol{k}\frac{3Qqa^2 v^2}{32\pi\varepsilon_0 r_0^3 c^2}\sin 2\theta_0. \tag{19}$$

2.3. *Torque on a uniformly charged sphere*

Since a thin disc may be regarded as a differential element of a sphere, we can find the torque acting on a small charged sphere of radius a by using equation (18). To do so, we replace in equation (18) a^4 by $(a^2 - y^2)^2$, replace τ by $\mathrm{d}y$ and integrate over y from $-a$ to $+a$. The result is

$$\boldsymbol{T} \approx \boldsymbol{k}\frac{3\rho q v^2}{32\varepsilon_0 r_0^3 c^2}\sin 2\theta_0 \int_{-a}^{+a}(a^4 - 2a^2y^2 + y^4)\,\mathrm{d}y \tag{20}$$

or

$$\boldsymbol{T} \approx \boldsymbol{k}\frac{\rho q a^5 v^2}{10\varepsilon_0 r_0^3 c^2}\sin 2\theta_0. \tag{21}$$

Replacing ρ in equation (21) by $3Q/4\pi a^3$, we find the torque acting on the sphere in terms of the charge Q of the sphere:

$$\boldsymbol{T} \approx \boldsymbol{k}\frac{3Qqa^2 v^2}{40\pi\varepsilon_0 r_0^3 c^2}\sin 2\theta_0. \tag{22}$$

3. Torque due to a point charge moving along a circular orbit

We start with equation (3) for the electric field of a point charge q moving with uniform velocity $\boldsymbol{v}$ along a circular orbit of radius R (figure 3). Let us find the electric field of q at the centre of the orbit. In this case $\boldsymbol{r} = \boldsymbol{R}$ and $\boldsymbol{r}\cdot\boldsymbol{v} = \boldsymbol{R}\cdot\boldsymbol{v} = 0$, so that equation (3) simplifies to

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 R^3}\left[\boldsymbol{R}\left(1 - \frac{v^2}{c^2}\right) - \boldsymbol{v}\frac{R}{c}\right]. \tag{23}$$

Equation (23) expresses the electric field in terms of the retarded position vector and retarded velocity vector of the charge. Retarded quantities are seldom observed in laboratories, and therefore, for practical applications, expressions containing retarded quantities should be converted to the present time (the time for which $\boldsymbol{E}$ is observed). We can convert equation (23)

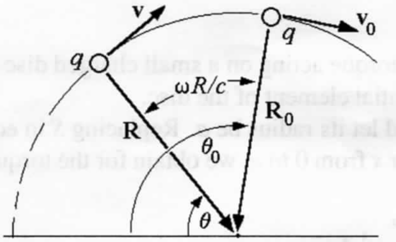

Figure 3. Correlation between the retarded quantities θ, $\boldsymbol{R}$ and $\boldsymbol{v}$ and the present-time quantities θ_0, $\boldsymbol{R}_0$ and $\boldsymbol{v}_0$ for a point charge q moving along a circular orbit.

to the present time by resolving the retarded position vector $\boldsymbol{R}$ and the retarded velocity vector $\boldsymbol{v}$ into their components along the present position vector $\boldsymbol{R}_0$ and the present velocity vector $\boldsymbol{v}_0$ (the radius R of the orbit is, of course, not affected by retardation and need not be converted). Since the angle between the present position vector and the retarded position vector is $\theta_0 - \theta = \omega R/c = v/c$, where ω is the angular velocity of the charge, we obtain for the two components of $\boldsymbol{E}$

$$E_{R_0} = \frac{q}{4\pi\varepsilon_0 R^3}\left[\left(1-\frac{v^2}{c^2}\right)R\cos(v/c) + \frac{Rv}{c}\sin(v/c)\right] \tag{24}$$

$$E_{v_0} = \frac{q}{4\pi\varepsilon_0 R^3}\left[\left(1-\frac{v^2}{c^2}\right)R\sin(v/c) - \frac{Rv}{c}\cos(v/c)\right] \tag{25}$$

and for the total field

$$\begin{aligned}\boldsymbol{E} = {} & \frac{q}{4\pi\varepsilon_0 R^3}\left[\left(1-\frac{v^2}{c^2}\right)\cos(v/c) + \frac{v}{c}\sin(v/c)\right]\boldsymbol{R}_0 \\ & + \frac{q}{4\pi\varepsilon_0 R^3}\left[\left(1-\frac{v^2}{c^2}\right)\frac{R}{v}\sin(v/c) - \frac{R}{c}\cos(v/c)\right]\boldsymbol{v}_0.\end{aligned} \tag{26}$$

In the calculations presented in section 2, we assumed that $v \ll c$. In the calculations that follow, we shall only assume that we can neglect v/c to powers higher than 3. Expanding $\sin(v/c)$ and $\cos(v/c)$ in equation (26) into power series of v/c and dropping terms containing v/c to powers higher than 3, we then obtain

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 R^3}\left[\left(1-\frac{v^2}{2c^2}\right)\boldsymbol{R}_0 - \frac{2Rv^2}{3c^3}\boldsymbol{v}_0\right]. \tag{27}$$

The $\boldsymbol{R}_0$ component of equation (27) is radially symmetric and therefore cannot contribute to the torque on a highly symmetrical charge Q at the centre of the orbit. Therefore, in the calculations that follow we only need to consider the $\boldsymbol{v}_0$ component of $\boldsymbol{E}$,

$$\boldsymbol{E}_{v_0} = -\frac{qv^2}{6\pi\varepsilon_0 R^2 c^3}\boldsymbol{v}_0. \tag{28}$$

3.1. Torque on a uniformly charged ring

Let q be negative and let it rotate about a positive-charged ring of uniform charge density ρ, cross sectional area, S and total charge Q whose centre is at the centre of the orbit and whose plane coincides with the plane of the orbit of q (figure 4). Let $\boldsymbol{v}$ satisfy the relation $v \ll c$ and

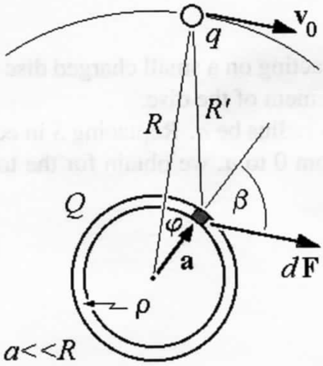

Figure 4. The torque acting on the ring carrying a charge Q and located at the centre of the orbit of q is found by integrating the torque acting on the shaded segment of the ring. Note that the force acting on the shaded sector is in the direction of the velocity vector $\boldsymbol{v}_0$ of q.

let the radius of the ring a satisfy the relation $a \ll R$. Although we have derived equation (28) for the centre of the orbit, it is approximately valid for points close to the centre, and since the radius of our ring is much smaller than the radius of the orbit, we can use equation (28) for finding an approximate expression for the torque exerted by q on the ring.

For the torque $\mathrm{d}\boldsymbol{T}$ with respect to the centre of the ring exerted by q on the charge element $\rho S a\,\mathrm{d}\phi$ contained in the shaded segment of the ring, we then have

$$\mathrm{d}\boldsymbol{T} = \rho S a\,\mathrm{d}\phi(\boldsymbol{a} \times \boldsymbol{E}_{v_0}) = \boldsymbol{k}\rho S a^2 E_{v_0} \sin\beta\,\mathrm{d}\phi = \boldsymbol{k}\rho S a^2 E_{v_0} \cos\phi\,\mathrm{d}\phi \qquad (29)$$

where the angles β and ϕ are as shown in figure 4, E_{v_0} is the magnitude of $\boldsymbol{E}_{v_0}$ at the location of the shaded segment, and $\boldsymbol{k}$ is a unit vector directed into the page. According to equation (28), taking into account that q is negative,

$$E_{v_0} = \frac{q v^3}{6\pi\varepsilon_0 R'^2 c^3} \qquad (30)$$

where R' is the distance between q and the shaded segment of the ring and v is the magnitude of the velocity vector $\boldsymbol{v}_0$ (which, of course, is the same as the magnitude of the velocity vector $\boldsymbol{v}$). From figure 4 we see that, since $a \ll R$, the distance from q to the shaded segment element of the ring is approximately

$$R' \approx R - a\cos\phi \qquad (31)$$

and therefore

$$\frac{1}{R'^2} \approx \frac{1}{(R - a\cos\phi)^2} \approx \frac{1}{R^2}\left(1 + \frac{2a}{R}\cos\phi\right). \qquad (32)$$

Substituting equations (30) and (32) into equation (29), we have

$$\mathrm{d}\boldsymbol{T} \approx \boldsymbol{k}\rho S a^2 \frac{q v^3}{6\pi\varepsilon_0 R^2 c^3}\left(1 + \frac{2a}{R}\cos\phi\right)\cos\phi\,\mathrm{d}\phi. \qquad (33)$$

Integrating equation (33) from 0 to 2π, we obtain for the torque acting on the ring

$$\boldsymbol{T} \approx \boldsymbol{k}\rho S a^3 \frac{q v^3}{6\varepsilon_0 R^3 c^3}. \qquad (34)$$

Replacing ρ in equation (34) by $Q/2\pi a S$, we obtain the expression for the torque in terms of the charge Q of the ring:

$$\boldsymbol{T} \approx \boldsymbol{k}\frac{q Q a^2 v^3}{12\pi\varepsilon_0 R^3 c^3}. \qquad (35)$$

3.2. Torque on a uniformly charged disc

We can use equation (34) for finding the torque acting on a small charged disc by considering the ring shown in figure 4 to be a differential element of the disc.

Let the thickness of the disc be τ and let its radius be a. Replacing S in equation (34) by $\tau\,dx$, replacing a by x and integrating over x from 0 to a, we obtain for the torque acting on the disc

$$\boldsymbol{T} \approx \boldsymbol{k}\frac{\rho\tau q v^3}{6\varepsilon_0 R^3 c^3}\int_0^a x^3\,dx \tag{36}$$

or

$$\boldsymbol{T} \approx \boldsymbol{k}\frac{\rho\tau a^4 q v^3}{24\varepsilon_0 R^3 c^3}. \tag{37}$$

Replacing ρ in equation (37) by $Q/\pi a^2\tau$, we find the torque acting on the disc in terms of the charge Q of the disc:

$$\boldsymbol{T} \approx \boldsymbol{k}\frac{qQa^2 v^3}{24\pi\varepsilon_0 R^3 c^3}. \tag{38}$$

3.3. Torque on a uniformly charged sphere

Since a thin disc may be regarded as a differential element of a sphere, we can find the torque acting on a small charged sphere of radius a by using equation (37). To do so, we replace a^4 in equation (37) by $(a^2 - y^2)^2$, replace τ by dy and integrate over y from $-a$ to $+a$. The result is

$$\boldsymbol{T} \approx \boldsymbol{k}\frac{\rho q v^3}{24\varepsilon_0 R^3 c^3}\int_{-a}^{+a}(a^4 - 2a^2y^2 + y^4)\,dy \tag{39}$$

or

$$\boldsymbol{T} \approx \boldsymbol{k}\frac{2\rho q a^5 v^3}{45\varepsilon_0 R^3 c^3}. \tag{40}$$

Replacing ρ in equation (40) by $3Q/4\pi a^3$, we find the torque acting on the sphere in terms of the charge Q of the sphere:

$$\boldsymbol{T} \approx \boldsymbol{k}\frac{qQa^2 v^3}{30\pi\varepsilon_0 R^3 c^3}. \tag{41}$$

4. Discussion

As we have seen, a moving charge does not merely attract or repel a stationary charge distribution, but also exerts a torque on it and thus causes it to rotate even if the stationary charge distribution is highly symmetric and has a uniform charge density. The direction of rotation depends on the polarity of the charges under consideration and on the direction of velocity of the moving charge.

In particular, when a negative point charge, starting from infinity, moves with constant speed along a straight line past a positive spherical charge, the point charge, as it comes closer to the spherical charge, exerts a torque on the spherical charge causing it to rotate so that the part of the spherical charge nearest to the point charge moves in the direction along which the point charge is moving. But then, as the point charge moves away from the spherical charge, the direction of the torque is reversed and, if not for the inertia of the spherical charge, it would rotate so that its part nearest to the point charge would move in the direction opposite to the direction along which the point charge is moving. According to equation (22), the torque is greatest at $\theta_0 = \pi/4$ and $\theta_0 = 3\pi/4$, and it is zero at $\theta_0 = 0$, $\theta_0 = \pi/2$ and $\theta_0 = \pi$.

If a negative point charge moves along a circular orbit around a positive spherical charge located at the centre of the orbit, the torque exerted by the point charge on the spherical charge is always in the same direction and causes the spherical charge to rotate in the same sense in which the point charge revolves.

Clearly, the dynamics of the interaction between a moving point charge and a stationary charge distribution is much more complicated than previously believed. The torque exerted by the moving charge on the stationary charge and the subsequent rotation of the stationary charge are only the initial stages of a very complex sequence of events. When the stationary charge rotates, it creates a magnetic field. In the case of a point charge moving along a straight line[2], the torque acting on the stationary charge is a function of time and therefore the angular velocity of the stationary charge is also a function of time. Therefore the magnetic field created by the stationary (rotating) charge is time dependent and, hence, induces an electric field. The induced electric field acts in turn on the moving point charge and affects its motion unless the motion is somehow controlled by external means. This is quite different from the attraction–repulsion interaction that we usually associate with a point charge moving past a stationary charge.

The interaction between an orbiting point charge and a spherical charge at the centre of the orbit is even more complex. In principle, such a system can be closed and need not depend on external forces for its stability. However, because of the torque acting on the central charge, the stability of the system is not at all certain. First, because, by equation (41), the torque exerted by the orbiting point charge is always present, the angular velocity of the central charge constantly accelerates. Therefore the magnetic field resulting from the rotation is always present and so is the induced electric field. Clearly, under these conditions the orbiting charge cannot move with constant speed, and the radius of the orbit cannot remain the same unless there exists some as yet undisclosed mechanism that keeps the speed and the radius constant (for the purpose of this discussion we ignore radiation by the orbiting charge). Furthermore, there is a problem with the conservation of angular momentum. In a closed system, the sum of the mechanical angular momentum and the electromagnetic angular momentum must remain the same at all times. This means that the magnetic field of the rotating charge, the magnetic field of the moving point charge, and the electric fields of the moving point charge and of the stationary (rotating) charge must at all times maintain a very precise balance.

In summary, then, the interaction between a moving point charge and a stationary charge is a very complex phenomenon, the details of which are yet to be determined. However, it is quite clear that by assuming that the electric interaction between moving and stationary charges is merely a Coulomb-type interaction (as is done, for example, in quantum mechanics and quantum electrodynamics [9]), one can obtain only approximate solutions of the problems involving moving and stationary charges.

We shall conclude this discussion by pointing out a possible relevance of the results of this paper to gravitational interactions. In 1893, Oliver Heaviside published a paper suggesting the existence of an analogy between time-dependent gravitational fields and electromagnetic fields [10]. The existence of such analogy follows also from general relativity theory [11]. In his paper, Heaviside derived a formula for the gravitational field of a point mass moving with

[2] It should be noted that because of the interaction between the two charges such a motion is impossible unless the point charge is by some external means constrained to maintain its velocity.

constant speed along a straight line. In modern notation Heavisides's formula is

$$g = -G\frac{m(1-v^2/c^2)}{r^3[1-(v^2/c^2)\sin^2\theta]^{3/2}}r \tag{42}$$

where g is the gravitational field, G is the gravitational constant, m is the mass of the moving point mass, v is the velocity of the mass, c is the speed of propagation of gravitation (generally assumed to be the same as the speed of light), r is the radius vector joining the moving point mass with the point of observation, r is the magnitude of r and θ is the angle between the velocity vector v and r. As one can see, except for the sign and symbols, the formula is the same as equation (1). Therefore all the consequences obtained above on the basis of equation (1) should be applicable to gravitational fields. Furthermore, it is clear that if equation (42) holds, then the field of an orbiting point mass should be, by analogy with equation (27),

$$g = -G\frac{m}{R^3}\left[\left(1-\frac{v^2}{2c^2}\right)R_0 - \frac{2Rv^2}{3c^3}v_0\right]. \tag{43}$$

Therefore an orbiting mass (planet or satellite) may be expected to exert a torque on the central body and cause this body to rotate. Of course, because of the factor v^3/c^3 in equation (28) (and therefore the same factor in the corresponding gravitational equation) and because planets and satellites move with speeds very much smaller than the speed of light, the torque acting on the central mass in a planetary system should be very small. However, taking into account that the timescale in cosmic systems is extremely long, the cumulative effect of the gravitational torque may be significant. It remains to be seen whether or not such a gravitational effect exists.

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