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Cylindrical Electrets

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Abstract

Cylindrical-shell electrets with radially-symmetric and axially-symmetric polarization are discussed. Theoretical expressions for the electric fields of such electrets are obtained. Possible applications of cylindrical electrets are indicated.

Introduction

Most published works on electrets deal with electrets in the shape of disks, plane-parallel slabs, and thin films. However, electrets of other shapes may be more appropriate for various experimental and theoretical studies. Spherical-shell electrets[1] were recently discussed and studied by Jefimenko and Sun (1972). The purpose of the present paper is to provide basic electric field data on cylindrical-shell electrets.

Theoretical

Consider a cylindrical-shell electret placed between two grounded conducting shields coaxial with the electret (Fig. 1). Let the electric field in the space between the inner shield and the electret be $\bar{E}_1$, the electric field in the electret be $\bar{E}_2$, and the electric field in the space between the electret and the outer shield be $\bar{E}_3$. We are interested in expressing $\bar{E}_1$, $\bar{E}_2$, and $\bar{E}_3$ as functions of characteristic geometrical and electrical parameters of the system. For a sufficiently long electret (such that the end effects of the system may be neglected) this can be done as follows.

Let the radii of the inner shield, inner electret surface, outer electret surface, and outer shield be a, b, c, and d, respectively. Since the shields are at the same potential, we have

$$\int_a^b \bar{E}_1\cdot d\bar{r} + \int_b^c \bar{E}_2\cdot d\bar{r} + \int_c^d \bar{E}_3\cdot d\bar{r} = 0. \qquad (1)$$

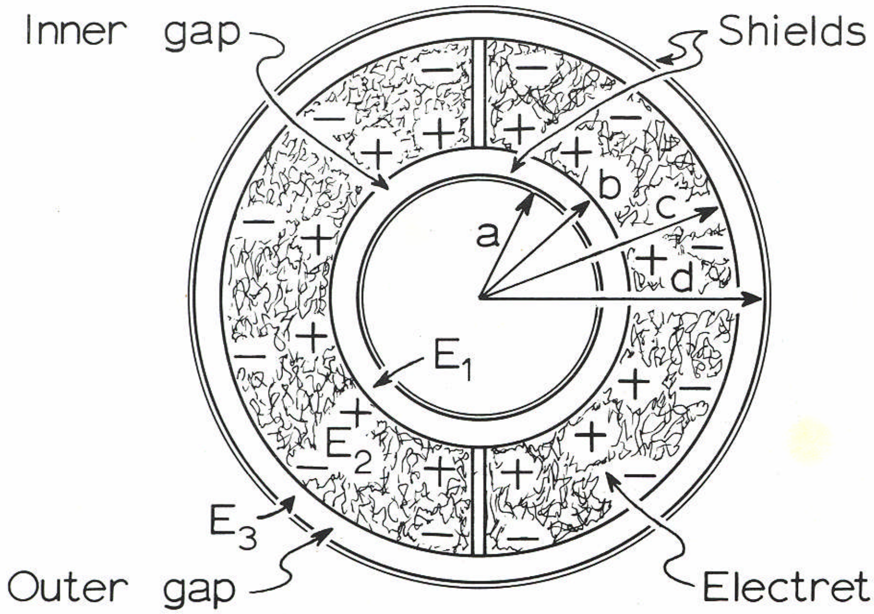

FIGURE 1. Radially-symmetric system.

At the surfaces of the electret the boundary conditions for the displacement vector $\overline{D}$ must be satisfied. Let the real surface charge densities on the inner and outer surfaces of the electret be σ_{12} and σ_{23}, respectively. At the outer surface we then have

$$(\overline{D}_3 - \overline{D}_2)\cdot\overline{r}_u = \sigma_{23}, \qquad r = c, \tag{2}$$

and at the inner surface we have

$$(\overline{D}_2 - \overline{D}_1)\cdot\overline{r}_u = \sigma_{12}, \qquad r = b, \tag{3}$$

where $\overline{r}_u$ is the radial unit vector, and the subscripts on the $\overline{D}$'s correspond to those on the $\overline{E}$'s.

The displacement vector inside the electret, $\overline{D}_2$, may be expressed in terms of the field vector $\overline{E}_2$ and the polarization vector $\overline{P}$ of the electret as

$$\overline{D}_2 = \epsilon_o\overline{E}_2 + \overline{P} , \tag{4}$$

where ϵ_o is the permittivity of space. Since there is no polarization outside the electret, $\overline{D}_3$ may be written as

$$\bar{D}_3 = \varepsilon_o \bar{E}_3. \tag{5}$$

Substituting Eqs. (4) and (5) into Eq. (2), we have

$$\varepsilon_o(\bar{E}_3 - \bar{E}_2)\cdot\bar{r}_u = \sigma_{23} + \bar{P}\cdot\bar{r}_u, \qquad r = c. \tag{6}$$

The electret polarization $\bar{P}$ may be assumed to be the sum of an induced polarization $\bar{P}_i$ (function of $\bar{E}_2$) and a remanent polarization $\bar{P}_r$ (independent of $\bar{E}_2$). For $\bar{P}_i$ we have

$$\bar{P}_i = \varepsilon_o(\varepsilon - 1)\bar{E}_2, \tag{7}$$

where ϵ is the permittivity of the electret. Assuming that the electret has been formed in a radially symmetric forming field (or otherwise possesses a radially symmetric remanent polarization), we can write for $\bar{P}_r$

$$\bar{P}_r = \frac{p}{r}\,\bar{r}_u\ , \tag{8}$$

where p is a constant. The total polarization in the electret is then

$$\bar{P} = \frac{p}{r}\,\bar{r}_u + \varepsilon_o(\varepsilon - 1)\bar{E}_2\ , \tag{9}$$

which, with Eq. (6), yields

$$\varepsilon_o(E_3 - \varepsilon E_2) = \sigma_{23} + \frac{p}{c}\ , \qquad r = c, \tag{10}$$

where we took into account the assumed absence of edge effects and the radial symmetry of the system under consideration in eliminating r_u on the left side of the equation.

A similar calculation employing Eq. (3) yields for the inner surface of the electret

$$\varepsilon_o(\varepsilon E_2 - E_1) = \sigma_{12} - \frac{p}{b}\ , \qquad r = b. \tag{11}$$

Combining Eqs. (1), (10), and (11) and once again making use of the radial symmetry of the system, we obtain after some elementary transformations

$$\bar{E}_1 = \frac{K_1}{r}\,\bar{r}_u, \qquad \bar{E}_2 = \frac{K_2}{r}\,\bar{r}_u, \qquad \bar{E}_3 = \frac{K_3}{r}\,\bar{r}_u, \tag{12}$$

where the constants K_1, K_2, and K_3 are given by

$$K_1 = -\frac{q_i(\varepsilon \ln \frac{d}{c} + \ln \frac{c}{b}) + q_o \varepsilon \ln \frac{d}{c}}{2\pi\ell\varepsilon_o[\varepsilon(\ln \frac{b}{a} + \ln \frac{d}{c}) + \ln \frac{c}{b}]}, \tag{13}$$

$$K_2 = \frac{q_i \ln \frac{b}{a} - q_o \ln \frac{d}{c}}{2\pi\ell\varepsilon_o[\varepsilon(\ln \frac{b}{a} + \ln \frac{d}{c}) + \ln \frac{c}{b}]}, \tag{14}$$

$$K_3 = \frac{q_o(\varepsilon \ln \frac{b}{a} + \ln \frac{c}{b}) + q_i \varepsilon \ln \frac{b}{a}}{2\pi\ell\varepsilon_o[\varepsilon(\ln \frac{b}{a} + \ln \frac{d}{c}) + \ln \frac{c}{b}]}, \tag{15}$$

where $q_i = 2\pi\ell(\sigma_{12}b - p)$ represents the effective charge of the inner surface of the electret, $q_o = 2\pi\ell(\sigma_{23}c + p)$ represents the effective surface charge of the outer surface of the electret, and ℓ is the length of the electret.

The above expressions for K's may be considerably simplified for various special cases. In particular, if the outer shield is in contact with the outer surface of the electret ($d = c$), we obtain for K_1

$$K_1 = -\frac{q_i \ln \frac{c}{b}}{2\pi\ell\varepsilon_o(\ln \frac{c}{b} + \varepsilon \ln \frac{b}{a})}. \tag{16}$$

This expression can be further transformed into

$$K_1 = -\frac{q_i \ln (1 + \frac{L}{b})}{2\pi\ell\varepsilon_o[\ln (1 + \frac{L}{b}) + \varepsilon \ln (1 + \frac{d_i}{a})]}, \tag{17}$$

or

$$K_1 = -\frac{q_i \ln \beta}{2\pi\ell\varepsilon_o[\ln \beta - \varepsilon \ln (1 + \alpha - \alpha\beta)]}, \tag{18}$$

where $d_i = b - a$ is the width of the inner gap (space between the inner shield and the inner surface of the electret), $L = c - b$ is the thickness of the electret, $\alpha = d_i/L$, and $\beta = c/b$. Likewise, if the inner shield is in contact with the inner surface of the electret ($a = b$), we obtain for K_3

$$K_3 = \frac{q_o \ln \frac{c}{b}}{2\pi\ell\varepsilon_o(\ln \frac{c}{b} + \varepsilon \ln \frac{d}{c})}. \tag{19}$$

This expression can be further transformed into

$$K_3 = \frac{q_o \ln (1 + \frac{L}{b})}{2\pi \ell \varepsilon_o [\ln (1 + \frac{L}{b}) + \varepsilon \ln (1 + \frac{d_o}{c})]} , \tag{20}$$

or

$$K_3 = \frac{q_o \ln \beta}{2\pi \ell \varepsilon_o [\ln \beta + \varepsilon \ln (1 + \gamma - \gamma/\beta)]} , \tag{21}$$

where $d_o = d - c$ is the width of the outer gap (space between the outer surface of the electret and the outer shield), L and β are as before, and $\gamma = d_o/L$.

For practical applications one usually wants to know the charges induced on the inner and outer shields by the electret. The density of the induced charge on the inner shield is $\sigma_i = \varepsilon_o E$ surface, or, according to Eqs. (12) and (18)

$$\sigma_i = \frac{q_i \ln \beta}{2\pi \ell a[\ln \beta - \varepsilon \ln (1 + \alpha - \alpha\beta)]} . \tag{22}$$

The total induced charge on the inner shield is therefore

$$Q_i = \frac{q_i \ln \beta}{\ln \beta - \varepsilon \ln (1 + \alpha - \alpha\beta)} . \tag{23}$$

This charge has its maximum value

$$Q_{i\ max} = q_i \tag{24}$$

when the radius of the inner shield is equal to the radius of the inner surface of the electret ($d_i = 0$, and $\alpha = 0$ in this case). Let us define the "reduced charge" Q_i^{*} as

$$Q_i^{*} = \frac{Q_i}{Q_{i\ max}} . \tag{25}$$

From Eqs. (23) and (24) we then have

$$Q_i^{*} = \frac{\ln \beta}{\ln \beta - \varepsilon \ln (1 + \alpha - \alpha\beta)} . \tag{26}$$

Let now the inner shield, rather than the outer shield be in contact with the electret. As it follows from Eqs. (12) and (21) by a reasoning similar to that used in deriving Eq. (23), the charge induced on the outer shield is then

$$Q_o = \frac{q_o \ln \beta}{\ln \beta + \varepsilon \ln (1 + \gamma - \gamma/\beta)} , \qquad (27)$$

This charge has its maximum value

$$Q_{o\ max} = q_o \qquad (28)$$

when the radius of the outer shield is equal to the radius of the outer surface of the electret ($d_o = 0$, and $\gamma = 0$ in this case). Let us define the "reduced charge" Q_o^{*} as

$$Q_o^{*} = \frac{Q_o}{Q_{o\ max}} . \qquad (29)$$

From Eqs. (27) and (28) we then have

$$Q_o^{*} = \frac{\ln \beta}{\ln \beta + \varepsilon \ln (1 + \gamma - \gamma/\beta)} . \qquad (30)$$

In a similar manner one can define "reduced electric fields" E_i^{*} and E_o^{*} (for example, E_i^{*} is the field at the surface of the inner shield of a given radius a divided by the field at the surface of the inner shield of radius b). As can be easily seen

$$E_i^{*} = Q_i^{*} \frac{b}{a} , \qquad (31)$$

and

$$E_o^{*} = Q_o^{*} \frac{c}{d} . \qquad (32)$$

Theoretical curves for Q_i^{*} and E_i^{*} representing Eqs. (26) and (31) are shown in Fig. 2 for an electret of dielectric constant $\varepsilon = 2.5$, $b = 2.54$ cm, and $c = 3.81$ ($\beta = 1.5$), which are the parameters of the electrets normally used in our laboratory. Theoretical curves for Q_o^{*} and E_o^{*} representing Eqs. (30) and (32) for the same electret are shown in Fig. 3.

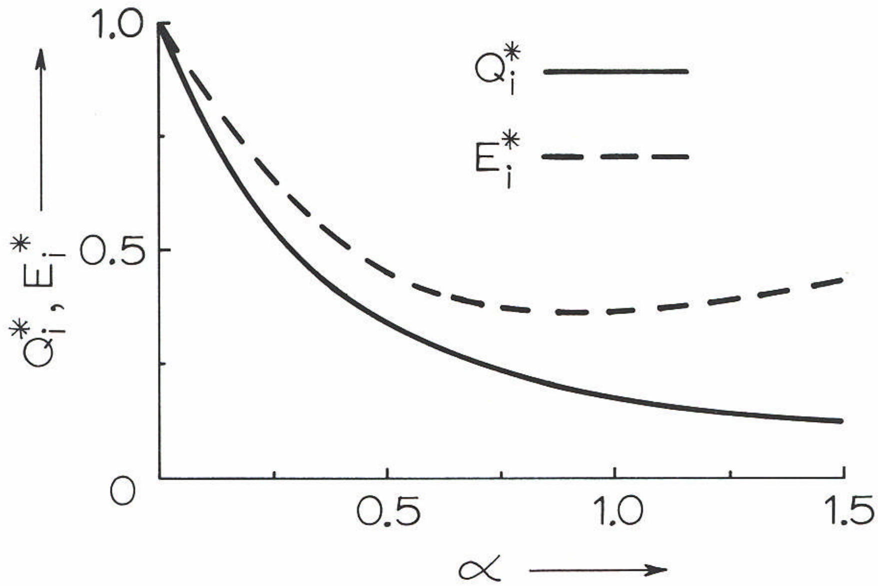

FIGURE 2. Reduced-charge and reduced-field curves for the inner shield.

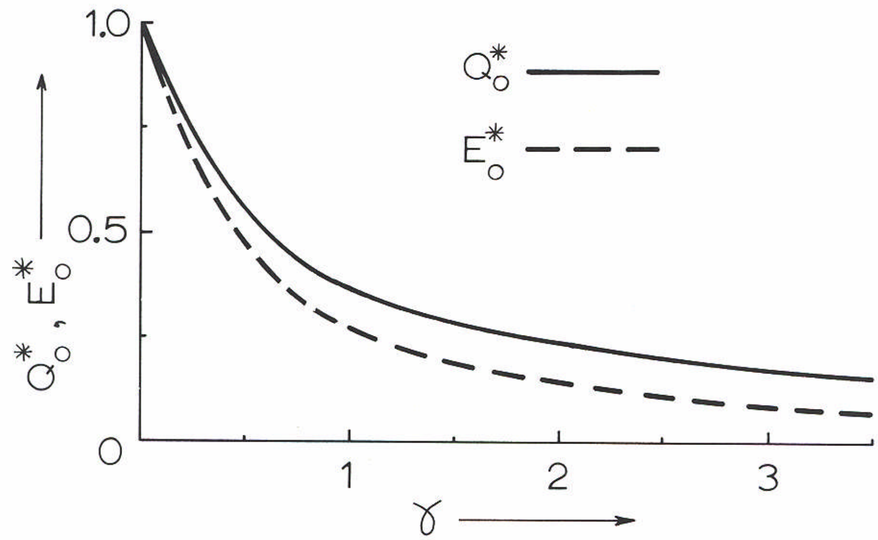

FIGURE 3. Reduced-charge and reduced-field curves for the outer shield.

The system shown in Fig. 1 was assumed to contain a radially-symmetric electret. Suppose now that this electret is replaced with an electret made of two halves of opposite polarity as shown in Fig. 4. The electric fields of such an axially-symmetric system can be found as follows.

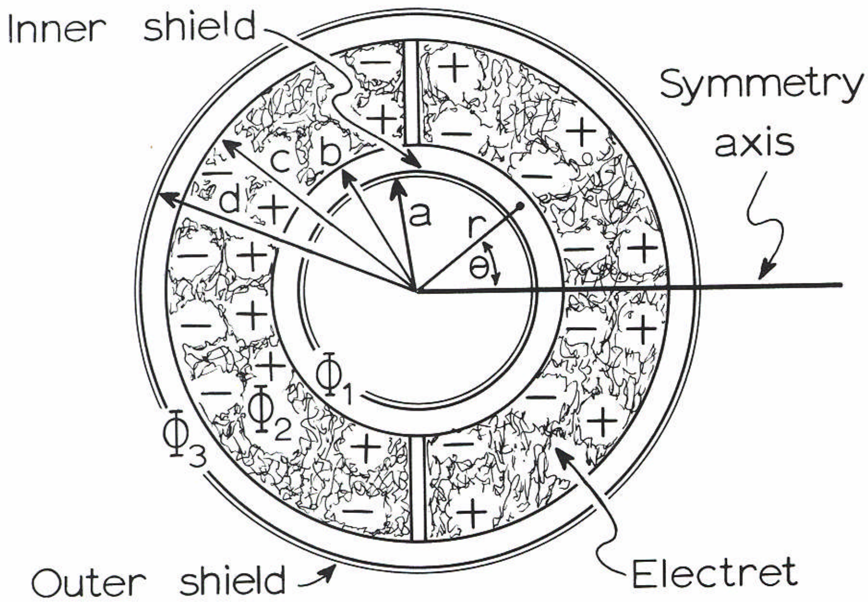

FIGURE 4. Axially-symmetric system.

Let the potentials in the inner gap, in the electret, and in the outer gap be φ_1, φ_2, and φ_3, respectively. From the symmetry of the system it follows that each of these potentials is of the form

$$\varphi = \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \cos n\theta, \tag{33}$$

where A_n and B_n are constants, n is an integer, and r and θ are cylindrical coordinates as shown in Fig. 4. The constants A_n and B_n can be found with the aid of the following boundary conditions:

$$\varphi_1 = 0 \quad \text{at} \quad r = a, \tag{34}$$

$$\varphi_3 = 0 \quad \text{at} \quad r = d, \tag{35}$$

$$\varphi_1 = \varphi_2 \quad \text{at} \quad r = b, \tag{36}$$

$$\varphi_2 = \varphi_3 \quad \text{at} \quad r = c, \tag{37}$$

$$(\bar{D}_2 - \bar{D}_1)\cdot \bar{r}_u = \sigma_{12} \quad \text{at} \quad r = b, \tag{38}$$

and

$$(\bar{D}_3 - \bar{D}_2)\cdot\bar{r}_u = \sigma_{23} \qquad \text{at} \qquad r = c. \tag{39}$$

In the system under consideration, especially interesting is the field in the inner gap or in the central cavity (when the inner shield is absent). For the special case when the outer shield is in contact with the electret (d = c) the coefficients A_{nl} and B_{nl} for the potential φ_1 are, as it follows from Eqs. (33)-(39) after somewhat lengthy calculations,

$$A_{n1} = \frac{L_n}{K_n}, \tag{40}$$

and

$$B_{n1} = -\frac{L_n}{K_n}\, a^{2n}, \tag{41}$$

where

$$L_n = -\frac{4\sigma_{ie}}{\pi\varepsilon_0 n}\sin\frac{n\pi}{2}, \tag{42}$$

$$K_n = nb^{n-1}\Big[1 + \varepsilon - (\varepsilon - 1)\,\frac{a^{2n}}{b^{2n}} - 2\varepsilon\,\frac{b^{2n} - a^{2n}}{b^{2n} - c^{2n}}\Big], \tag{43}$$

and $\sigma_{ie} = \sigma_{12} - p/b$ (the effective surface charge density of the inner surface of the electret). In the first approximation

$$\varphi_1 \approx (A_{11}r + B_{11}r^{-1})\cos\theta = \frac{L_1}{K_1}\Big(1 - \frac{a^2}{r^2}\Big)\, r\cos\theta, \tag{44}$$

where

$$L_1 = -\frac{4\sigma_{ie}}{\pi\varepsilon_0}, \tag{45}$$

and

$$K_1 = 1 + \varepsilon - (\varepsilon - 1)\,\frac{a^2}{b^2} - 2\varepsilon\,\frac{b^2 - a^2}{b^2 - c^2}. \tag{46}$$

It is interesting to note that for a = O (that is, when the inner shield is absent) Eq. (44) reduces to

$$\varphi \approx \frac{L_1}{K_1} r \cos\theta , \tag{47}$$

so that the field in the central cavity is then approximately homogeneous. The magnitude of this field is

$$E_1 \approx \frac{4\sigma_{1e}}{\pi\varepsilon_0 \left(1 + \varepsilon - 2\varepsilon \frac{b^2}{b^2 - c^2}\right)} . \tag{48}$$

Discussion

Cylindrical-shell electrets can be conveniently used as active elements for electrostatic electrometers, motors, generators, and charge dispensers. A comparison of the Q_1^a curve shown in Fig. 2 with the corresponding curve for spherical electrets[1] shows that the charge induced on the inner shield of a cylindrical electret decreases with increasing width of the inner gap slower than for a similar spherical electret. Therefor cylindrical electrets are preferable to spherical ones for devices utilizing charges induced on the inner shields. A comparison of Eq. (48) with the corresponding equation for spherical electrets[1] shows that the magnitude of the electric fields in the central cavities of cylindrical and similar spherical electrets are practically the same. However, since the central cavity of a cylindrical electret can be made as long as one pleases, cylindrical electrets are also preferable to spherical ones for devices utilizing electric fields in the central cavities of the electrets. One may expect therefore that cylindrical electrets will be incorporated in many electret devices and will be at least as useful for practical applications as the plane and the spherical electrets.

Literature Cited

1. Jefimenko, Oleg, and Chang, N. Y. Sun. 1973. Spherical Carnauba Wax Electrets, in *Electrets, Charge Storage, and Transport in Dielectrics.* The Electrochemical Society, New York.